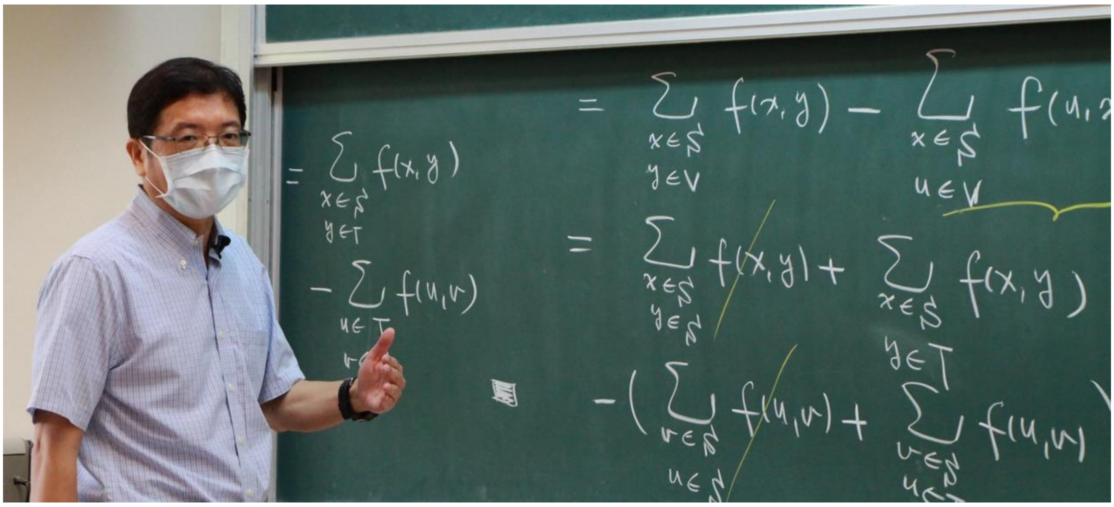
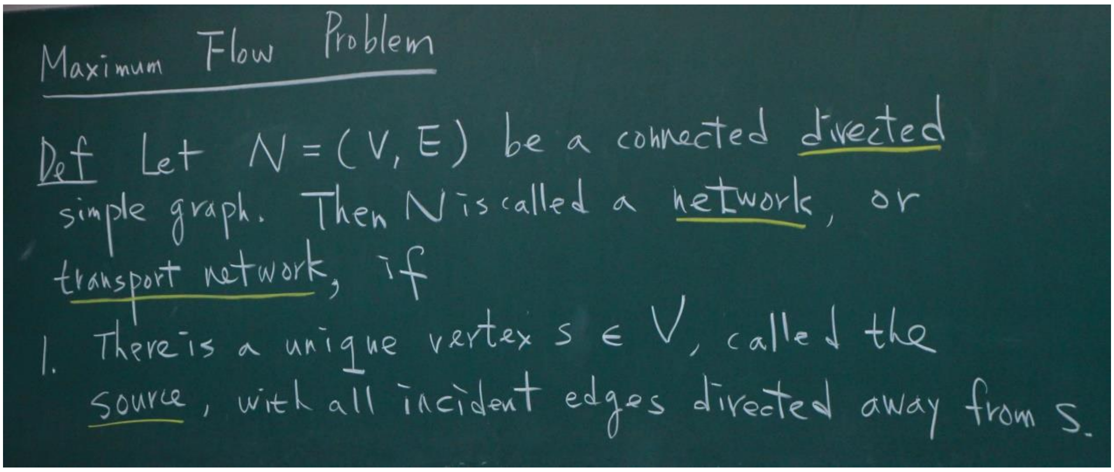
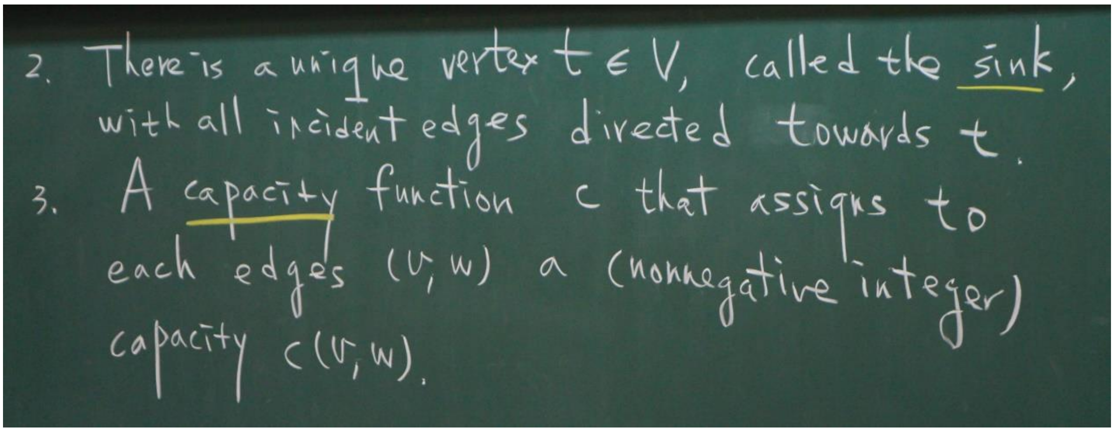
【10920 趙啟超教授離散數學 / 第 31 堂版書】







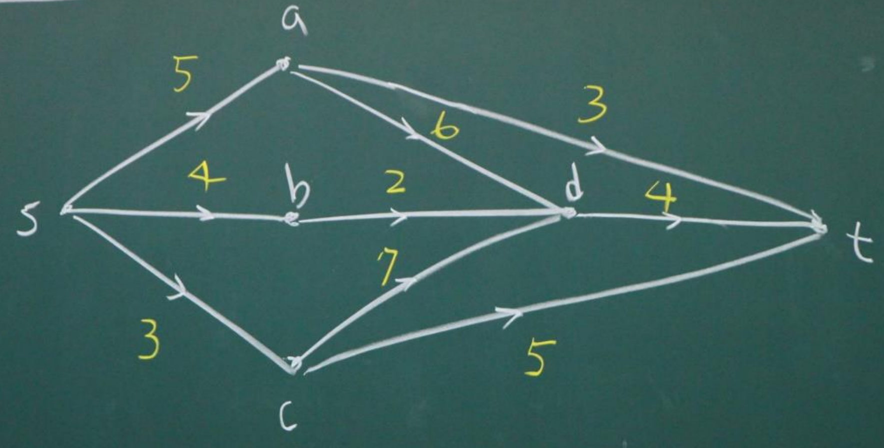
Maximum Flow Problem

Def Let $N = (V, E)$ be a connected directed simple graph. Then N is called a network, or transport network, if

1. There is a unique vertex $s \in V$, called the source, with all incident edges directed away from s.
2. There is a unique vertex $t \in V$, called the sink, with all incident edges directed towards t.
3. A capacity function c that assigns to each edges (v, w) a (nonnegative integer) capacity $c(v, w)$.

Example

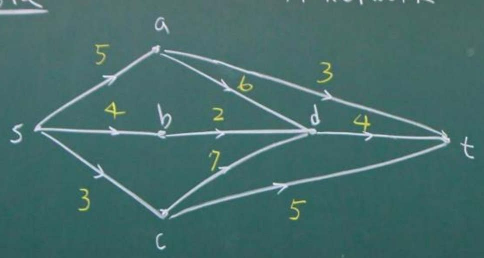
A network



Def A <u>flow</u> f in a network $N=(V,E)$ is a function that assigns to each edge e a nonnegative integer $f(e)$ such that

1. $f(e) \le c(e)$, for $e \in E$. (feasibility)
2. $\text{inflow}(v) = \sum_{u \in V} f(u,v) = \sum_{w \in V} f(v,w) = \text{outflow}(v)$ for $v \ne s,t$. (conservation)

Example A network



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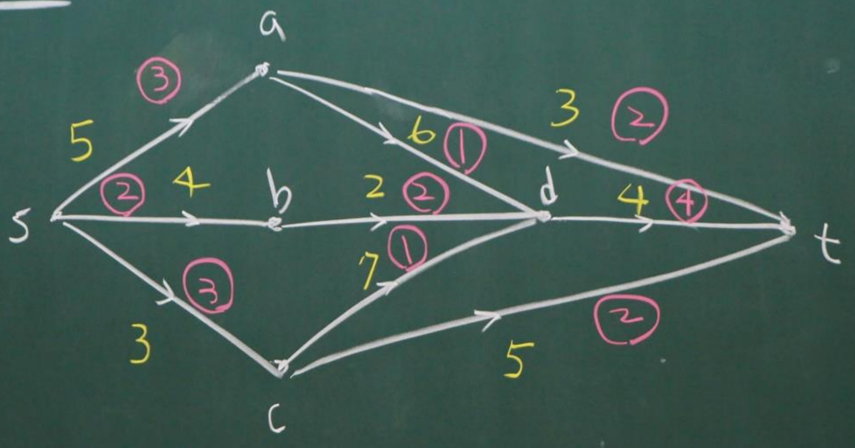

Remark If there is no edge (u,v), then
$$c(u,v) = f(u,v) = 0.$$

Example (Continued)

$\text{outflow}(s) = 3+2+3 = 8$

$\text{inflow}(a) = 3 = 1+2 = \text{outflow}(a)$

$\text{inflow}(b) = 2 = \text{outflow}(b)$

$\text{inflow}(c) = 3 = 1+2 = \text{outflow}(c)$

$\text{inflow}(d) = 1+2+1 = 4 = \text{outflow}(d)$

$\text{inflow}(t) = 2+4+2 = 8.$

Def The value of the flow is defined as the outflow of the source.

Example (Continued) $\text{val}(f) = \text{outflow}(s) = 8.$

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Example (Continued) $\text{val}(f) = \text{outflow}(s) = 8.$

Def A cut (S,T) (separating s and t) for a network $N=(V,E)$ is a partition of V: $V = S \cup T$ (with $S \cap T = \phi$) such that $s \in S$ and $t \in T$.

Property For a flow f and any cut (S, T),

$$\mathrm{val}(f) = \sum_{\substack{x \in S \\ y \in T}} f(x, y) - \sum_{\substack{u \in T \\ v \in S}} f(u, v)$$

Example (Continued)

Let $S = \{s, a, b, d\}$ and $T = \{c, t\}$.

Property For a flow f and any cut (S, T),

$$\mathrm{val}(f) = \sum_{\substack{x \in S \\ y \in T}} f(x, y) - \sum_{\substack{u \in T \\ v \in S}} f(u, v)$$

Example (Continued)

Let $S = \{s, a, b, d\}$ and $T = \{c, t\}$.

$$\sum_{\substack{x \in S \\ y \in T}} f(x, y) = 2+4+3 = 9.$$

$$\sum_{\substack{u \in T \\ v \in S}} f(u, v) = 1$$

$$\mathrm{val}(f) = 8 = 9 - 1.$$

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$$\mathrm{val}(f) = 8 = 9 - 1.$$

Proof $\quad \mathrm{val}(f) = \mathrm{outflow}(s) = \sum_{y \in V} f(s,y)$

$$= \sum_{y \in V} f(s,y) - \underbrace{\sum_{u \in V} f(u,s)}_{= \,\mathrm{inflow}(s) = 0}$$

$$= \sum_{y \in V} f(s,y) - \sum_{u \in V} f(u,s) + \sum_{x \in S - \{s\}} \Big(\underbrace{\sum_{y \in V} f(x,y) - \sum_{u \in V} f(u,x)}_{= \,\mathrm{outflow}(x) - \mathrm{inflow}(x) = 0} \Big)$$

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$$= \sum_{\substack{x \in S \\ y \in V}} f(x,y) - \underbrace{\sum_{\substack{x \in S \\ u \in V}} f(u,x)}_{= \sum_{\substack{v \in S \\ u \in V}} f(u,v)}$$

$$= \sum_{\substack{x \in S \\ y \in S}} \cancel{f(x,y)} + \sum_{\substack{x \in S \\ y \in T}} f(x,y) - \Big(\sum_{\substack{v \in S \\ u \in S}} \cancel{f(u,v)} + \sum_{\substack{v \in S \\ u \in T}} f(u,v) \Big)$$

$$= \sum_{\substack{x \in S \\ y \in T}} f(x,y) - \sum_{\substack{u \in T \\ v \in S}} f(u,v). \qquad \blacksquare$$

Proof

$$\mathrm{val}(f) = \mathrm{outflow}(s) = \sum_{y \in V} f(s,y)$$

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$$= \cancel{\sum_{\substack{x \in S \\ y \in S}} f(x,y)} + \sum_{\substack{x \in S \\ y \in T}} f(x,y) - \Big(\cancel{\sum_{\substack{v \in S \\ u \in S}} f(u,v)} + \sum_{\substack{v \in S \\ u \in T}} f(u,v) \Big)$$

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Remark Let $S = V - \{t\}$ and $T = \{t\}$.

We have $$\operatorname{val}(f) = \sum_{x \in V - \{t\}} f(x,t) - \sum_{v \in V - \{t\}} f(t,v)$$

$$= \operatorname{inflow}(t) - \underbrace{\operatorname{outflow}(t)}_{0}$$

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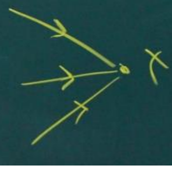

Def The capacity of the cut (S,T) is defined by $$\operatorname{cap}(S,T) = \sum_{\substack{x \in S \\ y \in T}} c(x,y).$$

Example (Continued)

If $S = \{s,a,b\}$ and $T = \{c,d,t\}$

$$\operatorname{cap}(S,T) = \sum_{\substack{x \in S \\ y \in T}} c(x,y) = 3+6+2+3 = 14.$$

If $S = \{s,b\}$ and $T = \{a,c,d,t\}$, $\operatorname{cap}(S,T) = \sum_{\substack{x \in S \\ y \in T}} c(x,y) = 5+2+3 = 10.$

Remark Let $S = V - \{t\}$ and $T = \{t\}$.

We have $\mathrm{val}(f) = \sum_{x \in V-\{t\}} f(x,t) - \sum_{v \in V-\{t\}} f(t,v)$

$= \mathrm{inflow}(t) - \underbrace{\mathrm{outflow}(t)}_{0}$

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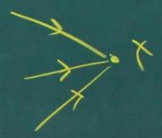

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Property For a flow f and any cut (S, T),
$$\mathrm{val}(f) \le \mathrm{cap}(S,T).$$

Proof
$$\mathrm{val}(f) = \sum_{\substack{x \in S \\ y \in T}} f(x,y) - \underbrace{\sum_{\substack{u \in T \\ v \in S}} f(u,v)}_{\ge 0}$$
$$\le \sum_{\substack{x \in S \\ y \in T}} f(x,y)$$
$$\le \sum_{\substack{x \in S \\ y \in T}} c(x,y) = \mathrm{cap}(S,T).$$

Remark max-flow $\leq$ min-cut

Example (Continued)

Consider the directed path s, a, t.
Neither the edge (s, a) nor (a, t) is carrying flow into its full capacity, and so we can increase the flow on both edges by $\min(5-3, 3-2) = 1$.
Hence $f_1(s, a) = 3+1 = 4$ and $f_1(a, t) = 2+1 = 3$.

The rules of feasibility and conservation still hold, and $\mathrm{val}(f_1) = 8+1 = 9$.
Now consider the "path" s, a, d, c, t.
A path in a network is a path in the associated undirected graph.

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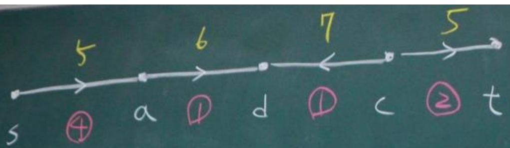

We can increase the flow on these edges by
$\min(5-4, 6-1, 1, 5-2) = 1$,
without violating the feasibility and conservation rules.

$$s \xrightarrow{5} a \xrightarrow{6} d \xleftarrow{7} c \xrightarrow{5} t$$

$$s \quad 4+1 \quad a \quad 1+1 \quad d \quad 1-1 \quad c \quad 2+1 \quad t$$

We than obtain a new flow f_2 with

$\mathrm{val}(f_2) = 9+1 = 10$.

Recall that if $S = \{s, b\}$ and $T = \{a, c, d, t\}$,

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$$s \xrightarrow{5} a \xrightarrow{6} d \xleftarrow{7} c \xrightarrow{5} t$$

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We than obtain a new flow f_2 with
$val(f_2) = 9+1 = 10$.
Recall that if $S=\{s,b\}$ and $T=\{a,c,d,t\}$,
$cap(S,T)=10$.

$\therefore$ f_2 is a maximum flow.

In general, given a flow f in a network, a path P from s to t is called an <u>f-augmenting path</u> provided that for each edge e on P,

$f(e) < c(e)$, for e a forward edge,
$f(e) > 0$, for e a backward edge.

Given an f-augmenting path p, the maximum amount of the flow we can increase on the forward edges and decrease on the backward edges, without violating the feasibility and conservation rules, is

$$\Delta_p = \min_{e \in p} \Delta_e,$$

where

$$\Delta_e = \begin{cases} c(e) - f(e), & \text{for } e \text{ a forward edge} \\ f(e), & \text{for } e \text{ a backward edge.} \end{cases}$$

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